DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91125

# STRATEGYPROOF AND NONBOSSY ASSIGNMENTS

Szilvia Papai

SOCIAL SCIENCE WORKING PAPER 938

January 1996

# Strategyproof and Nonbossy Assignments

Szilvia Papai


## Abstract

We consider the assignment of heterogeneous and indivisible objects to agents without using monetary transfers, where each agent may be assigned more than one object, and the valuation of the objects to an agent may depend on what other objects the agent is assigned. The set of strategyproof, nonbossy, and Pareto-optimal social choice functions is characterized as dictatorial sequential choice functions. Thus, the consequences of a Gibbard-Satterthwaite-type result can only be escaped in this context by using bossy social choice functions. It is also established that all strategyproof, strongly nonbossy and Pareto-optimal social choice functions are serial dictatorships, where strong nonbossiness is a stricter version of bossiness.

# Strategyproof and Nonbossy Assignments

Szilvia Papai*

## 1 Introduction

We consider the assignment of heterogeneous and indivisible objects to agents who have private information about their preferences. The objects are heterogeneous in the sense that they typically have different values to the agents. The agents may obtain any set of objects. Thus, our model is an extension of the much studied assignment model, where the agents can get at most one object. A most important characteristic of our model is that the objects may increase or decrease each other's values when obtained together. That is, the value of an object to an agent may not be independent of the other objects assigned to her. In addition, the agents cannot be charged for the objects, that is, monetary transfers are not allowed.

We require the allocation rules to be compatible with individual incentives. Thus, given that the agents have private information, the planner faces an implementation problem, a problem of designing an allocation mechanism that induces appropriate incentives for the agents. We examine allocation rules, called social choice functions, for which this implementation problem is solvable, using the dominant strategy solution concept. This solution concept requires the implementing mechanism to provide a best action (strategy) for each agent, regardless of the other agents' actions. In other words, the examined social choice functions are strategyproof.

When strategyproofness is required, attention is restricted to direct mechanisms, mechanisms that ask the agents to report their own preferences, due to the well-known revelation principle. Therefore, a direct mechanism that implements a social choice function will mirror the social choice function, in the sense that the outcome of the mechanism will coincide with the outcome prescribed by the social choice function for each preference profile. Thus, the criteria applied to the mechanisms apply to the social choice functions as well.

Dominant strategy equilibria are desirable, because they eliminate any strategic interaction among the agents. Admittedly, the existence of dominant strategy equilibria is

*I thank Kim C. Border, John O. Ledyard, Thomas R. Palfrey, and Simon Wilkie for helpful comments. Support from the Sloan Foundation is gratefully acknowledged.

a very strong requirement. Other common solution concepts, such as Nash-equilibrium and Bayesian-Nash-equilibrium are less demanding. The Nash-equilibrium concept, however, requires the agents to have full information about each other's preferences, while dominant strategy equilibria do not require any such information. The Bayesian-Nash solution concept, which also tends to produce better results, is based only on the knowledge of the agents' prior distributions. Nonetheless, the exact knowledge of these prior distributions is typically crucial to these results. Since dominant strategy mechanisms are robust in the sense that they do not use the information structure in the economy, it is essential to explore strategyproof mechanisms.

We also require social choice functions to be efficient. It is easy to see that a social choice function that prescribes the maximization of the sum of the utilities for agents is not strategyproof since only preference orderings can be elicited if strategyproofness is required.[1] Therefore, we use Pareto-optimality as a criterion of efficiency.

Another requirement we impose on social choice functions is nonbossiness. This criterion was introduced by Satterthwaite and Sonnenschein (1981), and used subsequently by Ritz (1983), Olson (1991), and Barbera and Jackson (1995), among others. A social choice function is bossy if there exists at least one agent whose preferences can change in a way that the prescribed allocation is different for some other agent(s), but not for herself, while everyone else's preferences are unchanged. Intuitively, this is an undesirable property, given that the mechanism mirrors the social choice function that it implements. This means that the agent who can change some other agent's allocation without changing her own may use her "power" by accepting a bribe or blackmailing. Admittedly, nonbossiness and strategyproofness together amount to more than individual incentive compatibility. Indeed, Barbera and Jackson (1995, Lemma 4) showed that these two conditions together imply a weak form of coalitional strategyproofness. However, while coalitional strategyproofness may be too restrictive for voting problems, it is also true that bossy behavior does not typically arise in these contexts. In fact, when indifferences are not admissable, bossiness obviously cannot occur, and ruling out indifferences might be quite agreeable in voting contexts. In contrast, when private goods are being allocated to selfish agents, indifferences cannot be ruled out.

The dominant strategy solution concept is indeed very demanding, which is illustrated by the famous Gibbard-Satterthwaite theorem. It states that in the context of voting the only social choice functions which induce truthful reporting of the preferences designate some favored voter who dictates the outcome. Underlying this impossibility theorem is the assumption that all conceivable preferences of the agents are admissable. We need to remark, however, that a similar impossibility result has been established for various restricted preference domains. For example, Barbera and Peleg (1990) proved this negative result for continuous preferences, and Zhou (1991a) for continuous and convex preferences.[2] When the allocation of private goods is considered, the outcomes have as many components as agents, each component representing the allotted bundle of private

[1] See, for example, Le Breton and Sen (1995) or Olson (1991).

[2] See also Sprumont (1995) that surveys some of the relevant literature.

goods for some agent. If we assume that the agents are *selfish*, i.e., that they only care about their own bundle of goods, then not all conceivable preferences are admissable. In particular, preferences other than indifference are ruled out between any two outcomes for any agent when the agent's component is the same in the two outcomes. Since the indivisible objects are heterogeneous and their valuations are interdependent, any valuation of the sets of objects are conceivable, so that further assumptions beyond selfishness need not be imposed on the preferences. We do assume, however, for simplicity's sake, that the agents cannot be indifferent between any two sets of objects, and between obtaining and not obtaining a set of objects. That is, indifference is ruled out for each agent between any two assignments where the agent's components in the two assignments are not identical. Since this domain does not contain either the universal domain investigated by Gibbard (1973) and Satterthwaite (1975) or the restricted (public) domains for which a Gibbard-Satterthwaite-type result was proved, an obvious question to ask is whether a Gibbard-Satterthwaite-type impossibility result would be obtained for it. Studies that examine strategyproofness in the context of allocating private goods focus on divisible goods, so that a further a priori structure (e.g., continuity, convexity, nonsatiation, etc.) is imposed on the preferences (see, for example, Zhou (1991b) and Barbera and Jackson (1995)). Thus, the results in these papers don't apply to our problem either.

A closely related result was produced by a line of research in social choice theory that investigated the existence of Arrow social welfare functions on the so called private alternatives domains, starting with Kalai and Ritz (1980). The largest private alternatives domain that they considered is identical to our domain of preferences. Continuing this research, Ritz (1983) established a a reciprocity result for private alternatives domains between Arrow-type social welfare functions and Gibbard-Satterthwaite-type social choice functions (in fact, he allows for social choice correspondences). Given Example 1 in Kalai and Ritz (1980), Theorem 3 of Ritz (1983) implies that the domain we investigate does not admit any rational, strategyproof, nonbossy, and nondictatorial social choice correspondence. It is important to point out that rationality is not a reasonable requirement in the context of private goods allocation problems, as it requires that for every set of outcomes the social choice function has to select the best element of an Arrow social welfare function. In voting or public goods contexts it is a reasonable criterion if we wish to take into account that some outcomes (alternatives) may not be feasible. In the context of private goods allocation problems, however, an outcome not being "feasible" in this sense means that some particular *distribution* of the fixed amounts of private goods is not feasible, which does not make much sense. When private goods are being allocated, feasibility problems are related to the *amount* of the private goods available for distribution. If the amount of the private goods is reduced then the outcome space "shrinks" accordingly. However, any particular distribution of the available amount of private goods should be treated as feasible, unless the focus is on some special circumstances that restrict the agents a priori to obtain at most a certain amount of the goods.

Returning to the question on a Gibbard-Satterthwaite-type impossibility in our context, the answer turns out to be positive. However, the consequences of such an impossibility result can only be escaped in our model by using bossy social choice functions.

That is, any strategyproof, nonbossy, and Pareto-optimal social choice function is dictatorial for the multi-object assignment problem. We need to point out that dictatorship is defined in a weaker sense here than is usual in voting contexts. Our dictator is not a dictator in the strong sense that, given any profile of the other agents, the dictator can "determine" the outcome, i.e., the allocations to the other agents as well.[3] This definition is more apropriate in the context of private goods allocation problems because the nonexistence of the conventional dictatorship is a very weak requirement. It would be ruled out by Pareto-optimality (or nonbossiness) alone.[4] This weaker definition, however, is in the spirit of the original definition of dictatorship (see Gibbard (1973), for example), in that a dictator can get her first choice regardless of the others' will. Thus, given the feasibility constraints, our dictator affects the outcomes of the other agents, which makes the distribution of power lopsided. However, since the dictator may be indifferent among outcomes that give her top choice to her, a dictatorial mechanism, as defined in this study, may take into account other agents' preferences as well.

Given that dictatorship is defined here in a weak sense, it follows that not all dictatorial mechanisms are strategyproof, nonbossy, and Pareto-optimal, unlike on the universal domain. Since in our context indifferences cannot be ruled out entirely, if the dictator is indifferent among outcomes that give her top allocation to her, which implies that some objects are available for allocation among the rest of the agents, then there is still room for manipulation and bossiness, and it is possible to get a Pareto-dominated outcome. Therefore, we need to characterize the set of strategyproof, nonbossy, and Pareto-optimal social choice functions. We prove that a social choice function is strategyproof, nonbossy, and Pareto-optimal if, and only if, it is a *dictatorial sequential choice function.* A dictatorial sequential choice function is one in which for each profile there exists an ordering of the agents such that the first agent in the ordering gets her favorite allocation, then, from the remaining objects, the second agent in the ordering gets her favorite allocation, etc., until we run out of either the objects or the agents. However, the ordering of the agents at the different profiles is not arbitrary. For each profile, the first agent in the ordering must be the same, hence these social choice functions are dictatorial. Moreover, the ordering of the rest of the agents may only vary at the different profiles as a function of the allocations of the preceeding agents.

If the ordering of the agents is fixed a priori, that is, if it is independent of the prefer-

[3]This is not to be confused with the distinction between weak and strong dictatorship in Muller and Satterthwaite (1986), which has to do with the feasible sets of alternatives, i.e., whether the agent is a dictator over a single feasible set or every feasible set. Satterthwaite (1975) distinguishes between fully and partially dictatorial voting procedures, which depends on whether it is in the dictator's power to impose any outcome or whether she is constrained to some subset of the possible outcomes. This is also different from the distinction discussed here.

[4]Zhou (1991b) proves for the two-agent case, where private goods are divisible and the admissable utility functions are continuous, strictly quasi-concave, and increasing, that any Pareto-optimal and strategyproof mechanism is *inversely dictatorial.* A mechanism is inversely dictatorial if one agent gets 0 at each profile. This is also a very weak requirement in our context, since the agents may have a negative evaluation for any set of objects, and thus Pareto-optimality alone ensures that an SCF is not inversely dictatorial.

ence profile, we get a *serial dictatorship*. Serial dictatorships were investigated for private goods economies by Satterthwaite and Sonnenschein (1981). In particular, they characterized the set of serial dictatorships in the context of allocating divisible private goods. Although they did not impose further restrictions on the domain beyond selfishness and a condition, called broad applicability, which requires that the set of admissible utility functions is open, a condition that amounts to certain "richness" of the domain, they required several differentiability conditions to ensure that the mechanism is "smooth." Clearly, their result does not apply to our model with indivisible goods. A further question is then, whether we need to impose much stricter criteria than strategyproofness, nonbossiness, and Pareto-optimality to get serial dictatorships. Interestingly, only nonbossiness needs to be strengthened to limit our choice exclusively to serial dictatorships. We call the more stringent criterion that replaces bossiness in this characterization result *strong nonbossiness*. It requires that the agents should only be able to affect each other's allocations through the feasibility constraints. It is, thus, essentially a negative result if we consider that strong nonbossiness is not an extreme requirement. Compared to Satterthwaite and Sonnenschein's chracterization, it seems like a somewhat surprising result as well. While strong nonbossiness rules out some of the above described "affect" relationships, one of Satterthwaite and Sonnenschein's criterion, called everywhere total, requires that each agent can affect any other agent at any profile. This apparent contradiction will be discussed in more detail later.

As a final remark before the formal presentation of the results, let us emphasize the importance of the feasibility constraints. If the same set of allocations were available to each agent, regardless of what the others get, there would be no conflict to solve. As opposed to voting, or a public goods economy, in this case each agent would get her favorite allocation. That is, for private goods allocation problems, the conflict stems from the fact that the amount of private goods available for allocation is fixed, i.e., from the scarcity of the resources expressed in the feasibility constraints.

## 2 Notation, Definitions, and Some Initial Results

There are $n \geq 2$ agents and $k \geq 2$ objects to be allocated among the agents. Let $N$ denote the set of $n$ agents and $K$ the set of $k$ objects. Let $\mathcal{K}$ denote the union of the power set of $K$ and the set consisting of a *null object*, $\{0\}$, which is an artificial "object" used for notational convenience. We will say that if an agent is not assigned any object, it is assigned a null object.

An *outcome* $x$ is an $n \times (2^k - 1)$ matrix, in which each element $x_a^i$ is defined by

$$x_a^i = \begin{cases} 1 & \text{if } a \text{ is assigned to agent } i \\ 0 & \text{otherwise,} \end{cases}$$

$\forall i \in N$, $\forall a \in \mathcal{K}$. To make the notation simple, we will write that $x^i = a$ when $x_a^i = 1$ and $x_b^i = 0$, $\forall b \in \mathcal{K}$, $b \neq a$. If agent $i$ is not assigned any set of objects as part of outcome $x$, then $x^i = 0$. We will refer to $x^i$ as agent $i$'s allocation in outcome $x$.

An outcome $x$ is *feasible* if none of the objects is assigned more than once, i. e., $\bigcap_{i \in N} x^i = \emptyset$ or $\{0\}$. Denote the set of feasible outcomes by $\mathcal{X}$.

Let $\theta^i_a$ denote the value that agent $i$ places on the set of objects denoted by $a$. Then $\theta^i = (\theta^i_1, \ldots, \theta^i_{2^k-1})$ is a list of the values placed by agent $i$ on the set of sets of objects, which we will refer to as *preferences*. The value of the null object is zero to each agent $i$ with any preferences $\theta^i$. We assume that each agent $i$ is selfish, that is, that each agent $i$ only cares about the $i$th element of $x$, which implies that she is indifferent between any two outcomes in which she gets the same allocation. For notational convenience, we define a utility function for each agent $i$ by $U(x^i, \theta^i) = \sum_{a \in \mathcal{K}} x^i_a \theta^i_a, \forall x \in \mathcal{X}, \forall i \in N$. We also assume that each agent has strict preferences over her allocations. That is, $\forall \theta^i \in \Theta^i$, $\theta^i_a \neq \theta^i_b$ whenever $a \neq b, \forall a, b \in \mathcal{K}$. Let $\Theta^i$ be the set of admissable preferences for agent $i$, so that $\theta^i \in \Theta^i$, $\forall i \in N$. The set of admissable preferences for all agents is denoted by $\Theta = \times_{i \in N} \Theta^i$. Let $\theta \in \Theta$ denote a profile of the agents. Similarly, let $\theta^{-i}$ be the profile of all the agents except for agent $i$.

**Definition 1** A *social choice function* is a function $f : \Theta \mapsto \mathcal{X}$.

**Definition 2** A *mechanism* $(g, S)$ is a set of strategy spaces $S_i$, $\forall i \in N$, where $S = \times_{i \in N} S_i$, and a function $g : S \mapsto \mathcal{X}$.

**Definition 3** A *direct mechanism* $g$ is a mechanism for which agent *i's* strategy space is $S_i = \Theta^i$, $\forall i \in N$, so that $S = \Theta$.

Let $f^i(\theta)$ denote the allocation prescribed to agent $i$ by $f$ at $\theta$, and let $g^i(\theta)$ denote $i$'s allocation resulting from mechanism $g$, when the reported profile is $\theta$.

**Definition 4** An SCF $f$ is *strategyproof* if $\forall \theta \in \Theta$, $\forall i \in N$, $\forall \tilde{\theta}^i \in \Theta^i$, $U(f^i(\theta), \theta^i) \geq U(f^i(\tilde{\theta}^i, \theta^{-i}), \theta^i)$. If $f$ is not strategyproof then it is *manipulable*. Then $\theta \in \Theta, i \in N$ and $\tilde{\theta}^i \in \Theta^i$ such that $U(f^i(\theta), \theta^i) < U(f^i(\tilde{\theta}^i, \theta^{-i}), \theta^i)$. We then say that agent $i$ can *manipulate* at $\theta$ via $\tilde{\theta}^i$.

**Definition 5** An SCF $f$ is *nonbossy* if $\forall \theta \in \Theta$, $\forall i, j \in N$, $\forall \tilde{\theta}^i \in \Theta^i$, if $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i}$ , then $f^j(\theta) = f^j(\tilde{\theta}^i, \theta^{-i}$. If $f$ is not nonbossy then it is *bossy*. Then $\exists \theta \in \Theta, i, j \in N$ *and* $\tilde{\theta}^i \in \Theta^i$ *such that* $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i}$ , and $f^j(\theta) \neq f^j(\tilde{\theta}^i, \theta^{-i}$. We then say that $i$ is *bossy* at $\theta$ versus $(\tilde{\theta}^i, \theta^{-i})$.

**Definition 6** An SCF $f$ is *Pareto-optimal* if $\forall \theta \in \Theta$, there does not exist $y \in \mathcal{X}$ such that $\forall i \in N, U(y^i, \theta^i) \geq U(f^i(\theta), \theta^i)$, and, for some $j \in N, U(y^j, \theta^j) > U(f^j(\theta), \theta^j)$.

Let $\text{top}(\theta^i)$ denote the top-ranked set of objects according to $\theta^i$. That is, $\forall i \in N, \forall \theta^i \in \Theta^i, \forall a \in \mathcal{K}, U(\text{top}(\theta^i), \theta^i) \geq U(a, \theta^i)$.

**Definition 7** An SCF $f$ is *nondictatorial* if there does not exist $i \in N$ such that $\forall\theta \in \Theta, f^i(\theta) = \text{top}(\theta^i)$. If $f$ is not nondictatorial then it is *dictatorial.* Then $\exists i \in N$ such that $\forall\theta \in \Theta, f^i(\theta) = \text{top}(\theta^i)$. We then say that $i$ is a *dictator* for $f$.

A useful result is that strategyproofness and nonbossiness together imply monotonicity.

**Definition 8** An SCF $f$ satisfies *monotonicity* if $\forall\theta, \tilde{\theta} \in \Theta$ such that $f(\theta) = x$, if $\forall y \in \mathcal{X}, \forall i \in N, U(x^i, \theta^i) \geq U(y^i, \theta^i) \Rightarrow U(x^i, \tilde{\theta}^i) \geq U(y^i, \tilde{\theta}^i)$ then $f(\tilde{\theta}) = x$.

**Lemma 1** [5] *A strategyproof and nonbossy SCF is monotonic.*

*Proof:* Suppose $f$ is strategyproof, nonbossy, and $\exists\theta, \tilde{\theta} \in \Theta$ such that $f(\theta) = x$ and $\forall y \in \mathcal{X}, \forall i \in N, U(x^i, \theta^i) \geq U(y^i, \theta^i) \Rightarrow U(x^i, \tilde{\theta}^i) \geq U(y^i, \tilde{\theta}^i)$. Let $f(\tilde{\theta}^1, \theta^{-1}) = z$. Then either $z = x$ or $z^1 \neq x^1$, by $f$'s nonbossiness. If $z^1 \neq x^1$ then strategyproofness implies that $U(x^1, \theta^1) > U(z^1, \theta^1)$. Then by assumption, $U(x^1, \tilde{\theta}^1) > U(z^1, \tilde{\theta}^1)$, given that $z^1 \neq x^1$. However, this contradicts $f$'s strategyproofness. Therefore, $z = x$, and $f(\tilde{\theta}^1, \theta^{-1}) = x$. Repeating the same argument for $i = 2, \ldots, n$, we get that $f(\tilde{\theta}) = x$, as required.□

Remark that strategyproofness alone is equivalent to the IPM property, as was shown by Dasgupta et al. (1978).[6] However, it is not equivalent to *strong positive association,* (SPA) on our domain, in contrast with the domain that consists of all strict preferences, for which the equivalence was shown by Muller and Satterthwaite (1977). In fact, SPA is equivalent to monotonicity so that SPA implies strategyproofness, but strategyproofness alone does not imply SPA on our domain. Thus, Lemma 1 underlines that on the examined domain strategyproofness and nonbossines together rule out the same sources of strategic behavior as strategyproofness alone on the usual (public) domains.

Pareto-optimality and bossiness are incompatible when there are only two agents, which is demonstrated below. Accordingly, the results that require Pareto-optimality in the following sections can be restated without the nonbossiness assumption for the two-agent case.

**Lemma 2** *If there are only two agents then a Pareto-optimal SCF is nonbossy.*

*Proof:* Let $n = 2$ and let $f$ be Pareto-optimal and bossy. Suppose agent 1 is bossy. Then $\exists\theta \in \Theta$ and $\tilde{\theta}^1 \in \Theta^1$ such that $f^1(\theta^1, \theta^2) = f^1(\tilde{\theta}^1, \theta^2)$ and $f^2(\theta^1, \theta^2) \neq f^2(\tilde{\theta}^1, \theta^2)$. Let $f(\theta^1, \theta^2) = x$ and $f(\tilde{\theta}^1, \theta^2) = y$, so that $x^1 = y^1$ and $x^2 \neq y^2$. Then either $U(x^2, \theta^2) > U(y^2, \theta^2)$ or $U(x^2, \theta^2) < U(y^2, \theta^2)$, and $x^1 \bigcap x^2 = \emptyset, x^1 \bigcap y^2 = \emptyset$, by feasibility. This implies that either $y$ or $x$ is not Pareto-optimal.□

[5] Essentially the same result is shown in Olson (1991, Lemma 8.11) and Barbera and Jackson (1995, Lemma 2), although both in a somewhat different setting, and using completely different terminology. The proof is given here for self-containment.

[6] For the correct version of IPM, see, for example, Laffont and Maskin (1982).

# 3 Characterization of Strategyproof, Nonbossy, and Pareto-optimal Social Choice Functions

First we prove that any strategyproof, nonbossy, and Pareto-optimal SCF is dictatorial.

**Proposition 1** *A strategyproof, nonbossy, and Pareto-optimal SCF is dictatorial.*

To prove this result we need to introduce some more definitions and provide two lemmas. Both lemmas and the definitions to follow are based on Barbera (1983), who proved the Gibbard-Satterthwaite theorem using the concept of pivotal voters.

A *reshuffling* of a preference ordering around an outcome $x$ is another preference ordering under which $x$ preserves the same relative position to all the other outcomes. Formally, for $\theta^i \in \Theta^i$ and $x \in \mathcal{X}, \tilde{\theta}^i \in \Theta^i$ is a reshuffling of $\theta^i$ around $x$ if $\forall y \in \mathcal{X}, U(x, \theta^i) \geq U(y, \theta^i) \Leftrightarrow U(x, \tilde{\theta}^i) \geq U(y, \tilde{\theta}^i)$. Let $r(x, \theta^i)$ denote the set of reshufflings of $\theta^i$ around $x$. Clearly, no agent can change the outcome of a strategyproof and nonbossy SCF $f$ at any profile by changing her reported preferences to a reshuffling around that outcome. This follows immediately from monotonicity, or can be verified directly by checking that if $f(\theta) \neq f(\tilde{\theta}^i, \theta^{-i})$ for some agent $i$, profile $\theta$, and $\tilde{\theta}^i \in r(f(\theta), \theta^i)$ then, since $f$ is nonbossy, agent $i$ can manipulate $f$ either at $\theta$ via $\tilde{\theta}^i$, or at $(\tilde{\theta}^i, \theta^{-i})$ via $\theta^i$.

Let $(\theta^i)^x$ denote the preferences obtained from $\theta^i$ when $x$ is ranked first, preserving the ordering of all the other outcomes in $\theta^i$. Similarly, let $(\theta^i)_x$ denote the preference ordering when $x$ is ranked last, $(\theta^i)^{x,y}$ when $x$ is ranked first and $y$ is ranked second, and $(\theta^i)^x_y$ when $x$ is ranked first and $y$ is ranked last.

For $\mathcal{Y} \subseteq \mathcal{X}, i \in N$, and $\theta^i \in \Theta^i$, let $c(\mathcal{Y}, \theta^i) = \{x \in \mathcal{Y} \mid \forall y \in \mathcal{Y}, U(x, \theta^i) \geq U(y, \theta^i)\}$ be the set of the best outcomes in $\mathcal{Y}$ for agent $i$ with preferences $\theta^i$. For $\mathcal{Y} \subseteq \mathcal{X}, i \in N$, and $\theta^i \in \Theta^i$, let $c^i(\mathcal{Y}, \theta^i) = x^i$ such that $U(x^i, \theta^i) \geq U(y^i, \theta^i), \forall y \in \mathcal{Y}$, where $x \in \mathcal{Y}$. Given that only strict preferences over allocations are admissable, $c^i(\mathcal{Y}, \theta^i)$ is a singleton for each agent $i$ and $\theta^i$. Since it will be clear in the following which SCF we refer to, $c(\mathcal{Y}, \theta^i)$ and $c^i(\mathcal{Y}, \theta^i)$ are not indexed for $f$, just as in other definitions to follow. Let $o(j, \theta^{-j})$ denote agent $j$'s *option set* for $\theta^{-j}$, i.. e., the set of outcomes that can be achieved by $j$ where the rest of the agents report $\theta^{-j}$.

The lemma to follow states that no agent can change the option set of any other agent at any profile, by changing her preferences to a reshuffling around the outcome of $f$ at that profile, provided $f$ is strategyproof and nonbossy.

**Lemma 3** *If an SCF $f$ is strategyproof and nonbossy, then $\forall \theta \in \Theta, \forall i, j \in N, \forall \tilde{\theta}^i \in r(f(\theta), \theta^i)$, $o(j, \theta^{-j}) = o(j, (\tilde{\theta}^i, \theta^{-i,j}))$.*

*Proof:* Let $f$ be strategyproof and nonbossy. Let $\theta \in \Theta$, $i, j \in N$, and $\tilde{\theta}^i \in r(f(\theta), \theta^i)$. We will show that $o(j, (\tilde{\theta}^i, \theta^{-i,j})) \subseteq o(j, \theta^{-j})$. Since $f(\tilde{\theta}^i, \theta^{-i}) = f(\theta)$, and $\theta^i \in r(f(\theta), \tilde{\theta}^i)$, a similar argument will prove that $o(j, \theta^{-j}) \subseteq o(j, (\tilde{\theta^i}, \theta^{-i,j}))$, which establishes the desired result.

Suppose $o(j, (\tilde{\theta^i}, \theta^{-i,j})) \not\subseteq o(j, \theta^{-j})$. Then $\exists y \in \mathcal{X}$ such that $y \in o(j, (\tilde{\theta^i}, \theta^{-i,j}))$ and $y \notin o(j, \theta^{-j})$. Let $f(\theta) = f(\tilde{\theta^i}, \theta^{-i}) = x$. Then $x \neq y$, since $x \in o(j, \theta^{-j})$. Since $x = c(o(j, \theta^{-j}), \theta^j)$ and $y \notin o(j, \theta^{-j})$, we have $x = c(o(j, \theta^{-j}), (\theta^j)^y) = f((\theta^j)^y, \theta^{-j})$. However, $y \in o(j, (\tilde{\theta^i}, \theta^{-i,j}))$ implies that $c(o(j, (\tilde{\theta}^{-i}, \theta^{-i,j}), (\theta^j)^y) = y$, so $f(\tilde{\theta^i}, (\theta^j)^y, \theta^{-i,j}) = y$. In sum, we have $f(\theta^i, (\theta^j)^y, \theta^{-i,j}) = x$, and $f(\tilde{\theta^i}, (\theta^j)^y, \theta^{-i,j}) = y$. If $x^i \neq y^i$, then $\theta^i \in r(x, \tilde{\theta^i})$ implies that agent $i$ can manipulate either at $(\theta^i, (\theta^j)^y, \theta^{-i,j})$ via $\tilde{\theta^i}$ or at $f(\tilde{\theta^i}, (\theta^j)^y, \theta^{-i,j})$ via $\theta^i$. This contradicts $f$'s strategyproofness, hence $x^i = y^i$. However, in this case nonbossiness implies that $x = y$, which is a contradiction.□

The next lemma is about the agents' ability to affect each other's allocation. An agent *affects* another agent at a given profile if she can change the other agent's allocation by deviating her messages.

**Definition 9** For an SCF $f$, agent $i$ *affects* agent $j$ at $\theta \in \Theta$, if $\exists \tilde{\theta^i} \in \Theta^i$ such that $f^j(\theta) \neq f^j(\tilde{\theta^i}, \theta^{-i})$. We then write that $iA(\theta)j$.

The following lemma states that if two agents can affect one another at some profile, then at least one of them is able to "get" the allocation the other one "imposes" on her, by deviating her message at that profile, or the "imposed" allocation is the null object for at least one of them, given a strategyproof and nonbossy SCF.

**Lemma 4** *If an SCF $f$ is strategyproof and nonbossy then $\forall \theta \in \Theta, \forall i, j \in N, i \neq j$, such that $iA(\theta)j$ and $jA(\theta)i$, and $\forall \tilde{\theta^i} \in \Theta^i, \forall \tilde{\theta^j} \in \Theta^j$ such that $f^j(\tilde{\theta^i}, \theta^{-i}) \neq f^j(\theta)$ and $f^i(\tilde{\theta^j}, \theta^{-j}) \neq f^i(\theta)$, we have one of four cases: (a) $f^j(\tilde{\theta^i}, \theta^{-i}) = f^j(\tilde{\theta^j}, \theta^{-j})$, (b) $f^i(\tilde{\theta^j}, \theta^{-j}) = f^i(\tilde{\theta^i}, \theta^{-i})$, (c) $f^j(\tilde{\theta^i}, \theta^{-i}) = 0$, or (d) $f^i(\tilde{\theta^j}, \theta^{-j}) = 0$.*

*Proof:* Let $f$ be strategyproof and nonbossy. Let $\theta \in \Theta, i, j \in N, i \neq j$, such that $iA(\theta)j$ and $jA(\theta)i$. Fix $\tilde{\theta^i} \in \Theta^i$ such that $f^j(\tilde{\theta^i}, \theta^{-i}) \neq f^j(\theta)$, and fix $\tilde{\theta^j} \in \Theta^j$ such that $f^i(\tilde{\theta^j}, \theta^{-j}) \neq f^i(\theta)$. Let $f(\theta) = x, f(\tilde{\theta^i}, \theta^{-i}) = y$, and $f(\tilde{\theta^j}, \theta^{-j}) = z$. Then $x, y \in o(i, \theta^{-i})$, $x, z, \in o(j, \theta^{-j})$, $z^i \neq x^i$, and $y^j \neq x^j$. Suppose $y^j \neq z^j, y^i \neq z^i, y^j \neq 0$, and $z^i \neq 0$. Since $z^i \neq 0$, it is possible that $U(x^i, \theta^i) < U(z^i, \theta^i)$, and, similarly, since $y^j \neq 0$ it is possible that $U(x^j, \theta^j) < U(y^j, \theta^j)$. Then we can define $\bar{\theta^i}, \bar{\theta^j}, \hat{\theta^i}$, and $\hat{\theta^j}$ as follows. Let

$$\bar{\theta^i} = \begin{cases} (\theta^i)_{z^i} & \text{if} \quad U(x^i, \theta^i) > U(z^i, \theta^i) \\ (\theta^i)^{z^i} & \text{if} \quad U(x^i, \theta^i) < U(z^i, \theta^i), \end{cases}$$

$$\bar{\theta^j} = \begin{cases} (\theta^j)_{y^j} & \text{if} \quad U(x^j, \theta^i) > U(y^j, \theta^j) \\ (\theta^j)^{y^j} & \text{if} \quad U(x^j, \theta^j) < U(y^j, \theta^j), \end{cases}$$

$$\hat{\theta^i} = \begin{cases} (\theta^i)^{y^i}_{z^i} & \text{if} \quad U(x^i, \theta^i) > U(z^i, \theta^i) \\ (\theta^i)^{z^i, y^i} & \text{if} \quad U(x^i, \theta^i) < U(z^i, \theta^i), \end{cases}$$

and

$$\hat{\theta}^j = \begin{cases} (\theta^j)^{z^j}_{y^j} & \text{if} \quad U(x^j,\theta^i) > U(y^j,\theta^j) \\ (\theta^j)^{y^j,z^j} & \text{if} \quad U(x^j,\theta^j) < U(y^j,\theta^j). \end{cases}$$

Note that $\bar{\theta}^i \in r(x^i,\theta^i), \bar{\theta}^j \in r(x^j,\theta^j), \hat{\theta}^i \in r(z^i,\bar{\theta}^i)$, and $\hat{\theta}^j \in r(y^j,\bar{\theta}^j)$.

Since $\bar{\theta}^j \in r(x^j,\theta^j), o(i,(\bar{\theta}^j,\theta^{-i,j})) = o(i,\theta^{-i})$, by Lemma 3. Then $y \in o(i,(\bar{\theta}^j,\theta^{-i,j}))$, so $c(o(i,(\bar{\theta}^j,\theta^{-i,j}),\hat{\theta}^i) = y$ if $U(x^i,\theta^i) > U(z^i,\theta^i)$. If $U(x^i,\theta^i) < U(z^i,\theta^i)$ then $f(\theta) \neq z$ indicates that $z \notin o(i,\theta^{-i})$, and so $z \notin o(i,(\bar{\theta}^j,\theta^{-i,j}))$. Thus, if $U(x^i,\theta^i) < U(z^i,\theta^i)$, we also have $c(o(i,(\bar{\theta}^j,\theta^{-i,j}),\hat{\theta}^i) = y$. Therefore, $f(\hat{\theta}^i,\bar{\theta}^j,\theta^{-i,j}) = y$. Using a similar argument for agent $j$, we can show that $f(\bar{\theta}^i,\hat{\theta}^j,\theta^{-i,j}) = y$. But then, given that $\hat{\theta}^i \in r(z^i,\bar{\theta}^i)$ and $\hat{\theta}^j \in r(y^j,\bar{\theta}^j)$, we get that $f(\hat{\theta}^i,\hat{\theta}^j,\theta^{-i,j}) = y = z$, which is a contradiction.□

*Proof of Proposition 1:*

Let $f$ be strategyproof, nonbossy, and Pareto-optimal.

**Step 1:** Identification of the dictator.

Let $\theta^i \in \begin{pmatrix} K \\ 0 \end{pmatrix}, \forall i \in N$. Then Pareto-optimality implies that there exists an agent, say agent 1, who gets $K$ at $\theta$. That is, given the feasibility constraints, $f(\theta) = (K,0,\ldots,0)$.

**Step 2:** No agent can affect the dictator at a profile where each agent's first choice is $K$ and second choice is 0.

Let $\bar{\theta}^1 \in (0)$. Then $\exists i \in N \setminus \{1\}$ such that $f^i(\bar{\theta}^1,\theta^{-1}) = K$, by Pareto-optimality. Let this agent be agent 2, so that $f(\bar{\theta}^1,\theta^{-1}) = (0,K,0,\ldots,0)$, by feasibility. Then $1A(\theta)2$. Suppose $2A(\theta)1$. Then, by Lemma 4, we have one of three cases: (a) $\exists\tilde{\theta}^2 \in \Theta^2$ such that $f^2(\tilde{\theta}^2,\theta^{-2}) = K$, or (b) $\exists\tilde{\theta}^2 \in \Theta^2$ such that $f^1(\tilde{\theta}^2,\theta^{-2}) = f^1(\bar{\theta}^1,\theta^{-1}) = 0$, or (c)$f^2(\bar{\theta}^1,\theta^{-1}) = 0$. Clearly, (c) doesn't hold. If (a) holds then agent 2 can manipulate at $\theta$ via $\tilde{\theta}^2$. If (b) holds then Pareto-optimality implies that either agent 2 gets $K$ at $(\tilde{\theta}^2,\theta^{-2})$, which leads to the same contradiction as in case (a), or some agent other than 1 or 2 gets $K$ at $(\tilde{\theta}^2,\theta^{-2})$, which implies that agent 2 is bossy. Therefore, $\neg(2A(\theta)1)$.

Next, we show that $\forall i \in N \setminus \{1,2\}, \neg(iA(\theta)1)$. Fix $i \in N \setminus \{1,2\}$. Suppose $iA(\theta)1$. Then $\exists\tilde{\theta}^i \in \Theta^i$ such that $f^1(\tilde{\theta}^i,\theta^{-i}) \neq f^1(\theta)$. By nonbossiness, $f^i(\tilde{\theta}^i,\theta^{-i}) \neq f^i(\theta) = 0$. We know that $f^i(\tilde{\theta}^i,\theta^{-i}) \neq K$, otherwise agent $i$ can manipulate at $\theta$ via $\tilde{\theta}^i$. Therefore, $f^i(\tilde{\theta}^i,\theta^{-i}) = p$, where $p \in \mathcal{K}, p \neq K, p \neq 0$. Then feasibility and Pareto-optimality imply that $f^j(\tilde{\theta}^i,\theta^{-i}) = 0, \forall j \in N \setminus \{i\}$. Now let $\hat{\theta}^i \in \begin{pmatrix} K \\ p \end{pmatrix}$. Then Pareto-optimality implies that $\exists j \in N$ such that $f^j(\hat{\theta}^i,\theta^{-i}) = K$. If $f^i(\hat{\theta}^i,\theta^{-i}) = K$, then $i$ can manipulate at $\theta$ via $\hat{\theta}^i$. If $f^j(\hat{\theta}^i,\theta^{-i}) = K$ for some $j \in N \setminus \{1,i\}$ then $i$ is bossy at $(\hat{\theta}^i,\theta^{-i})$ versus $\theta$. Therefore, $f^1(\hat{\theta}^i,\theta^{-i}) = K$, and the feasibility constraints imply that $f^i(\hat{\theta}^i,\theta^{-i}) = 0$. However, in this case, agent $i$ can manipulate at $(\hat{\theta}^i,\theta^{-i})$ via $\tilde{\theta}^i$, which contradicts $f$'s strategyproofness. This completes the proof that $\forall i \in N \setminus \{1\}, \neg(iA(\theta)1$.

**Step 3:** No coalition of the $n-1$ non-dictators can change the outcome, as long as the dictator's first choice is $K$.

Given Step 2, no agent other than 1 can change the outcome at $\theta = \begin{pmatrix} K & \cdots & K \\ 0 & \cdots & 0 \end{pmatrix}$, by changing her strategy alone. Now we want to show that no coalition of the $n-1$ agents, excluding agent 1, can change the outcome at $\theta$ by jointly deviating. Assume the contrary. Then $\exists \tilde{\theta}^{-1} \in \Theta^{-1}$ such that $f(\theta^1, \tilde{\theta}^{-1}) \neq (K, 0, \ldots, 0)$. If $n = 2$, then Step 3 holds by Step 2, so let $n \geq 3$. Let $\tilde{\Theta}^{-1} \subset \Theta^{-1}$ be a subset of the set of preference profiles for the $n-1$ agents, such that $\forall \tilde{\theta}^{-1} \in \tilde{\Theta}^{-1}, f(\theta^1, \tilde{\theta}^{-1}) \neq (K, 0, \ldots, 0)$. For all $\tilde{\theta}^{-1} \in \tilde{\Theta}^{-1}$, let $L(\tilde{\theta}^{-1}) = \left\{ i \in N \setminus \{1\} \mid \tilde{\theta}^i \notin \begin{pmatrix} K \\ 0 \end{pmatrix} \right\}$. Let $l = \min_{(\tilde{\theta}^{-1}) \in \tilde{\Theta}^{-1}} \{ \mid L(\tilde{\theta}^{-1}) \mid \}$, i.e., $l$ is the minimum number of the agents contained in any coalition in $N \setminus \{1\}$ that can jointly change the outcome at $\theta$ by deviating their strategies. Note that $l \geq 2$, by Step 2.

Now fix $\tilde{\theta}^{-1} \in \tilde{\Theta}^{-1}$ such that $L(\tilde{\theta^{-1}}) = l$. Let $L = \left\{ i \in N \setminus \{1\} \mid \tilde{\theta}^i \notin \begin{pmatrix} K \\ 0 \end{pmatrix} \right\}$, and let $f(\theta^1, \tilde{\theta}^{-1}) = f\begin{pmatrix} K, & \tilde{\theta}^L, & K & \cdots & K \\ 0 & & 0 & \cdots & 0 \end{pmatrix} = \mathrm{x}$ , assuming, without loss of generality, that $\forall i \in L, i \leq l+1$. Then monotonicity implies that $f\begin{pmatrix} K & x^L & K & \cdots & K \\ 0 & & 0 & \cdots & 0 \end{pmatrix} = x$, where $x^L = (x^2, \ldots, x^{l+1})$. Given that $\mid L \mid = l$, $f\begin{pmatrix} K, & \theta^i, & x^{L \setminus \{i\}} & K & \cdots & K \\ 0 & & & 0 & \cdots & 0 \end{pmatrix} = (K, 0, \ldots, 0), \forall i \in L$, where $\theta^i \in \begin{pmatrix} K \\ 0 \end{pmatrix}$. Since $x \neq (K, 0, \ldots, 0)$, $x^1 = 0$, by Pareto-optimality and feasibility. Now let $\bar{L} = N \setminus (L \bigcup \{1\})$, so that $N$ can be partitioned into $\{1\}, L$, and $\bar{L}$. (Note that $\bar{L} = \emptyset$ if $l = n-1$.) We know that $x^{\bar{L}} = (x^{l+2}, \ldots, x^n) = (0, \ldots, 0)$, otherwise some $j \in \bar{L}$ gets $K$ at $(\theta^1, \tilde{\theta}^{-1})$, and thus each $i \in L$ is bossy at $(\theta^1, \tilde{\theta}^{-1})$ versus $(\theta^1, \theta^i, \tilde{\theta}^{-1,i})$, given the feasibility constraints. We also know that $\exists i^* \in L$ such that $x^{i^*} \neq 0$, otherwise Pareto-optimality requires that either $x^1 = K$ or $x^j = K$ for some $j \in \bar{L}$, which is a contradiction. But then $\forall i \in L \setminus \{i^*\}, x^i \neq 0$, otherwise $i$ is bossy at $(\theta^1, \tilde{\theta}^{-1})$ versus $(\theta^1, \theta^i, \tilde{\theta}^{-1,i})$. Therefore, $\forall i \in L, x^i \neq 0$. Given that $\mid L \mid \geq 2$ and $\forall i \in L, x^i \neq 0$, the feasibility constraints imply that $x^i \neq K, \forall i \in L$. Therefore, $\mid L \mid \leq k$.

Now we will show that

$$
\begin{aligned}
&f\begin{pmatrix} K & K & x^3 & \cdots & x^{l+1} & K & \cdots & K \\ 0 & x^2 & & & & 0 & \cdots & 0 \end{pmatrix} = \\
&f\begin{pmatrix} K & K & \cdots & K & x^{i+1} & \cdots & x^{l+1} & K & \cdots & K \\ 0 & x^2 & \cdots & x^i & & & & 0 & \cdots & 0 \end{pmatrix} = \\
&f\begin{pmatrix} K & K & \cdots & K & K & \cdots & K \\ 0 & x^2 & \cdots & x^{l+1} & 0 & \cdots & 0 \end{pmatrix} = x.
\end{aligned} \tag{1}
$$

First notice that no agent other than 1 can get $K$, as long as agent 1 and each agent $j \in \bar{L}$ report $\begin{pmatrix} K \\ 0 \end{pmatrix}$, since otherwise some agent $i \in L$ is bossy, given the feasibility constraints. (If $\mid L \mid= 2$ and one agent in $L$ gets $K$ then the other agent in $L$ is the bossy agent. ) If the outcome were $(K, 0, \ldots, 0)$ for any of the above preference profiles then the appropriate agent $i$ ($i \in L$) can manipulate via $(x^i)$. Therefore, Pareto-optimality implies that (1) holds.

Using monotonicity, we get that $f\begin{pmatrix} 0 & K & \cdots & K & K & \cdots & K \\ & x^2 & \cdots & x^{l+1} & 0 & \cdots & 0 \end{pmatrix} = x$. Now take $i^* \in L$ such that

$$f^{i^*}\begin{pmatrix} 0 & K & \cdots & K & 0 & \cdots & 0 \\ & 0 & \cdots & 0 & & & \end{pmatrix} = K, \tag{2}$$

where $\forall i \in L$, $i$'s strategy is $\begin{pmatrix} K \\ 0 \end{pmatrix}$, and $\forall i \notin L$, $i$'s strategy is $(0)$. Note that $i^*$ satisfying (2) exists by Pareto-optimality. Let $i^* = 2$, without loss of generality. If agent 2 gets $K$ at

$$\begin{pmatrix} 0 & K & K & \cdots & K & 0 & \cdots & 0 \\ & 0 & x^3 & \cdots & x^{l+1} & & & \end{pmatrix}$$

then agent 2 can manipulate at

$$\begin{pmatrix} 0 & K & \cdots & K & 0 & \cdots & 0 \\ & x^2 & \cdots & x^{l+1} & 0 & \cdots & 0 \end{pmatrix}$$

via $\begin{pmatrix} K \\ 0 \end{pmatrix}$. If some other $i \in L, i \neq 2$ gets $K$ at that profile, then monotonicity implies that (2) is contradicted. Therefore,

$$f\begin{pmatrix} 0 & K & K & \cdots & K & 0 & \cdots & 0 \\ & 0 & x^3 & \cdots & x^{l+1} & & & \end{pmatrix} = (0, 0, x^3, \ldots, x^{l+1}, 0, \ldots, 0),$$

by Pareto-optimality. Then monotonicity implies that

$$f\begin{pmatrix} 0 & 0 & K & \cdots & K & 0 & \cdots & 0 \\ & & x^3 & \cdots & x^{l+1} & & & \end{pmatrix} = (0, 0, x^3, \ldots, x^{l+1}, 0, \ldots, 0).$$

Now let $L^2 = L \setminus \{2\}$, and apply the same argument to $L^2$ as the one applied to $L$ above. Letting $i^* = 3$, where $i^* \in L^2$ satisfies

$$f^{i^*}\begin{pmatrix} 0 & 0 & K & \cdots & K & 0 & \cdots & 0 \\ & & 0 & \cdots & 0 & & & \end{pmatrix} = K,$$

we get that

$$f\begin{pmatrix} 0 & 0 & K & K & \cdots & K & 0 & \cdots & 0 \\ & & 0 & x^4 & \cdots & x^{l+1} & & & \end{pmatrix} =$$

$$f\begin{pmatrix} 0 & 0 & 0 & K & \cdots & K & 0 & \cdots & 0 \\ & & & x^4 & \cdots & x^{l+1} & & & \end{pmatrix} = (0, 0, 0, x^4, \ldots, x^{l+1}, 0, \ldots, 0).$$

Continuing iteratively until we get to $L^{l-1}$, we find that

$$f^{l+1}\begin{pmatrix} 0 & \cdots & 0 & K & K & 0 & \cdots & 0 \\ & & & 0 & x^{l+1} & & & \end{pmatrix} = x^{l+1},$$

which violates Pareto-optimality. Note that we can get this contradiction for any number of agent in $L$, as long as $\mid L \mid \geq 2$, and regardless of the size of $\bar{L}$, which might be the empty set. Furthermore, since $2 \leq\mid L \mid\leq k$, we need at least two objects. Therefore, this proof applies to any number of agents such that $n \geq 3$ and any number of objects such that $k \geq 2$.

Therefore, $\forall\tilde{\theta}^{-1} \in \Theta^{-1}, f^1(\theta^1, \tilde{\theta}^{-1}) = K$, where $\theta^1 \in (K)$. But then $\forall\tilde{\theta}^1 \in \Theta^1$ such that $\tilde{\theta}^1 \in \begin{pmatrix} K \\ 0 \end{pmatrix}, \forall\tilde{\theta}^{-1} \in \Theta^{-1}, f^1(\tilde{\theta}) = K$, by monotonicity, which is what we wanted to show.□

**Step 4:** No coalition of the $n-1$ non-dictators can change the outcome for the dictator at any profile.

Let $\tilde{\theta}^1 \in \begin{pmatrix} p \\ K \\ 0 \end{pmatrix}$, where $p \in \mathcal{K}, p \neq K, p \neq 0$. Suppose $f^1(\tilde{\theta}) \neq p$ for some $\tilde{\theta}^{-1} \in \Theta^{-1}$. Then $f^1(\tilde{\theta}) = K$, otherwise Step 3 implies that agent 1 can manipulate at $\tilde{\theta}$ via $\theta^1 \in (K)$. However, in this case $f(\tilde{\theta}) = (K, 0, \ldots, 0)$, given the feasibility constraints, and thus the outcome $(p, 0, \ldots, 0)$ Pareto-dominates $(k, \ldots, 0)$ at $\tilde{\theta}$. Therefore, Pareto-optimality implies that $f^1(\tilde{\theta}) = p$. Then, by monotonicity, $\forall\tilde{\theta}^1 \in (p), \forall\tilde{\theta}^{-1} \in \Theta^{-1}, f^1(\tilde{\theta}) = p$. Finally, if $\tilde{\theta}^1 \in (0)$ then $f^1(\tilde{\theta}) = 0, \forall\tilde{\theta}^{-1} \in \Theta^{-1}$, by Pareto-optimality. Thus, together with Step 3, we have $\forall\tilde{\theta} \in \Theta, f^1(\tilde{\theta}) = \text{top}(\tilde{\theta}^1)$. Therefore, agent 1 is a dictator, and $f$ is dictatorial.□

In order to get an analog of the Gibbard-Satterthwaite theorem for nonbossy mechanisms, we show that a strategyproof and nonbossy SCF that satisfies *citizen sovereignty* is Pareto-optimal.

**Definition 10** An SCF $f$ satisfies *citizen sovereignty (CS)* if $\forall x \in \mathcal{X}, \exists\theta \in \Theta$ such that $f(\theta) = x$.

**Proposition 2** *A strategyproof, nonbossy, and CS SCF is Pareto-optimal.*

*Proof:* Let $f$ be strategyproof, nonbossy, CS, and not Pareto-optimal. Then $\exists x, y \in \mathcal{X}$ with $f(\theta) = x$ for some $\theta \in \Theta$, such that $\forall i \in N, U(y^i, \theta^i) \geq U(x^i, \theta^i)$, and for some $j \in N, U(y^j, \theta^j) > U(x^j, \theta^j)$. Define $\tilde{\theta} \in \Theta$ as follows. For each $i \in N$ such that $x^i \neq y^i$, let $\tilde{\theta}^i \in \begin{pmatrix} y^i \\ x^i \end{pmatrix}$, and for each $i \in N$ such that $x^i = y^i$, let $\tilde{\theta}^i \in (y^i)$. Then $f(\tilde{\theta}) = x$, by monotonicity. Since $f$ is CS, $\exists\bar{\theta} \in \Theta$ such that $f(\bar{\theta}) = y$. Now let $\hat{\theta}^i \in (y^i)$. Then $f(\hat{\theta}) = y$, by monotonicity. However, $\tilde{\theta}^i \in r(y, \hat{\theta}^i), \forall i \in N$ so that $x = y$. This is a contradiction, since $U(y^j, \theta^j) > U(x^j, \theta^j)$, for some $j \in N$.□

**Corollary 1** *A strategyproof, nonbossy, and CS SCF is dictatorial.*

The corollary follows directly from Propositions 1 and 2.

Not all dictatorial mechanisms are strategyproof, nonbossy, and Pareto-optimal since in our context indifferences cannot be ruled out entirely, and we defined dictatorship accordingly. Therefore, if the dictator is indifferent among outcomes that give her top allocation to her, which implies that some objects are available for allocation among the rest of the agents (at least one), then there is still room for manipulation and bossiness, and it is possible to get a Pareto-dominated outcome. In the next proposition, we characterize the set of strategyproof, nonbossy, and Pareto-optimal SCF's.

Let $\Sigma(N)$ denote the set of permutations of $N$. Then $\sigma \in \Sigma(N)$ is an ordered list of the agents, i.e., $\sigma = (\sigma^1, \ldots, \sigma^n)$. For the following definition, let the null object be defined as the empty set, i.e., let $0 = \emptyset$. Let $\sigma : \Theta \mapsto \Sigma(N)$ be a function that assigns an ordered list of the agents to each profile. With a slight abuse of notation, we denote $\sigma(\theta)$ by $\sigma_\theta$ so that $\sigma_\theta = (\sigma_\theta^1, \ldots, \sigma_\theta^n), \forall \theta \in \Theta$. Then, if $\sigma_\theta^i = j$, we write that $\sigma_\theta(j) = i$.

For the following definition, let the null object be defined as the empty set, i.e., let $0 = \emptyset$.

**Definition 11** An SCF $f$ is a *sequential choice function* if $\exists \sigma : \Theta \mapsto \Sigma(N)$ such that $\forall \theta \in \Theta, f^{\sigma_\theta^1}(\theta) = c^{\sigma_\theta^1}(\mathcal{K}, \theta^{\sigma_\theta^1}) =$ top $(\theta^{\sigma_\theta^1})$, and, for $j \in N \setminus \{1\}, f^{\sigma_\theta^j}(\theta)$ are defined recursively by $f^{\sigma_\theta^j}(\theta) = c^{\sigma_\theta^j}(\mathcal{K} \setminus \bigcup_{i=1}^{j-1}\{f^{\sigma_\theta^i}(\theta)\}, \theta^{\sigma_\theta^j})$. We then call $\sigma_\theta$ an *s-hierarchy* associated with $f$ at $\theta$.

**Definition 12** An SCF $f$ is a *dictatorial sequential choice function* if it is a sequential choice function such that $\forall \theta, \tilde{\theta} \in \Theta, \sigma_\theta^1 = \sigma_{\tilde{\theta}}^1$, and, $\forall j \in N \setminus \{1\}$, if $f^{\sigma_\theta^i}(\theta) = f^{\sigma_{\tilde{\theta}}^i}(\tilde{\theta})$ for $i = 1, \ldots, j-1$, then $\sigma_\theta^j = \sigma_{\tilde{\theta}}^j$.

**Proposition 3** *An SCF is strategyproof, nonbossy, and Pareto-optimal if, and only if, it is a dictatorial sequential choice function.*

*Proof:*

(a) First we prove that a dictatorial sequential choice function is strategyproof, nonbossy, and Pareto-optimal. It is easy to verify that a sequential choice function is Pareto-optimal, hence we will only show i) strategyproofness and ii) nonbossiness.
i) Let $f$ be a dictatorial sequential choice function. First we show that an agent cannot change her rank in the appropriate orderings by deviating alone. Fix $\theta \in \Theta, j \in N$, and $\tilde{\theta}^j \in \Theta^j$. Let $\sigma_\theta(j) = t$ and $\sigma_{(\tilde{\theta}^j, \theta^{-j})}(j) = l$, where $t, l \in N$. Suppose $t \neq l$. If $t = 1$ then $l = 1$, so $t \neq l$ implies that $t \neq 1$. By symmetry, $l \neq 1$. Suppose $t = 2$. Then, since $t \neq 1$, and $l \neq 1$, $f^{\sigma_\theta^1}(\theta) = f^{\sigma^1_{(\tilde{\theta}^j, \theta^{-j})}}(\tilde{\theta}^j, \theta^{-j}) = \text{top}(\theta^{\sigma_\theta^1})$, which implies that $t \neq 2$, and by symmetry, $l \neq 2$. Continuing iteratively, we get that $t \notin N$, which is a contradiction. Therefore, $\forall \theta \in \Theta, \forall j \in N, \forall \tilde{\theta}^j \in \Theta^j, \sigma_\theta(j) = \sigma_{(\tilde{\theta}^j, \theta^{-j})}(j)$.

Now keep $\theta \in \Theta, j \in N$, and $\tilde{\theta}^j \in \Theta^j$ fixed and let $\sigma_\theta(j) = \sigma_{(\tilde{\theta}^j,\theta^{-j})}(j) = t$, where $t \in N$. Clearly, $j$ cannot manipulate if $t = 1$. If $t = 2$ then $\theta^{\sigma^1_\theta} = \theta^{\sigma^1_{(\tilde{\theta}^j,\theta^{-j})}}$ implies that $f^{\sigma^1_\theta}(\theta) = f^{\sigma^1_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j, \theta^{-j}) = \text{top}(\theta^{\sigma^1_\theta})$. Then $f^j(\theta) = c^j(\mathcal{K} \setminus \{\text{top}(\theta^{\sigma^1_\theta}\}, \theta^j)$ and $f^j(\tilde{\theta}^j, \theta^{-j}) = c^j(\mathcal{K} \setminus \{\text{top}(\theta^{\sigma^1_\theta}\}, \tilde{\theta}^j)$, so that $j$ cannot manipulate. Similarly, if $t > 2$, then $\theta^{\sigma^1_\theta} = \theta^{\sigma^1_{(\tilde{\theta}^j,\theta^{-j})}}$ implies that $f^{\sigma^1_\theta}(\theta) = f^{\sigma^1_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j, \theta^{-j}) = \text{top}(\theta^{\sigma^1_\theta})$, which in turn implies that $\sigma^2_\theta = \sigma^2_{(\tilde{\theta}^j,\theta^{-j})}$. Then $\theta^{\sigma^2_\theta} = \theta^{\sigma^2_{(\tilde{\theta}^j,\theta^{-j})}}$, which implies that $f^{\sigma^2_\theta}(\theta) = f^{\sigma^2_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j, \theta^{-j})$, etc, till we get to $t-1$. In sum, $f^{\sigma^i_\theta}(\theta) = f^{\sigma^i_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j, \theta^{-j})$, for $i = 1, \ldots, t-1$, and so $\mathcal{K} \setminus \bigcup_{i=1}^{t-1}\{f^{\sigma^i_\theta}(\theta)\} = \mathcal{K} \setminus \bigcup_{i=1}^{t-1}\{f^{\sigma^i_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j, \theta^{-j})\}$. Therefore, agent $j$ cannot manipulate for any $t \in N$. Since $\theta, j$, and $\tilde{\theta}^j$ were chosen arbitrarily, this proves that $f$ is strategyproof.

ii) Fix $\theta \in \Theta, j \in N$, and $\tilde{\theta}^j \in \Theta^j$. Then $\sigma_\theta(j) = \sigma_{(\tilde{\theta}^j,\theta^{-j})}(j)$ and $f^{\sigma^i_\theta}(\theta) = f^{\sigma^i_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j, \theta^{-j})$ for $i = 1, \ldots, t-1$, where $\sigma_\theta(j) = t$, by i). Suppose $f^j(\theta) = f^j(\tilde{\theta}^j, \theta^{-j})$. Then $\sigma^{t+1}_\theta = \sigma^{t+1}_{(\tilde{\theta}^j,\theta^{-j})}$, and $\mathcal{K} \setminus \bigcup_{i=1}^{t}\{f^{\sigma^i_\theta}(\theta)\} = \mathcal{K} \setminus \bigcup_{i=1}^{t}\{f^{\sigma^i_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j, \theta^{-j})\}$, which implies that $f^{\sigma^{t+1}_\theta}(\theta) = f^{\sigma^{t+1}_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j, \theta^{-j})$. This, in turn, implies that $\sigma^{t+2}_\theta = \sigma^{t+2}_{(\tilde{\theta}^j,\theta^{-j})}$. Continuing iteratively, we get that $f^{\sigma^l_\theta}(\theta) = f^{\sigma^l_{(\tilde{\theta}^j,\theta^{-j})}}(\tilde{\theta}^j, \theta^{-j})$ for $l = t+1, \ldots, n$, which proves that $f$ is nonbossy.

(b) Conversely, we prove that a strategyproof, nonbossy, and Pareto-optimal SCF is a dictatorial sequential choice function. Suppose $f$ is strategyproof, nonbossy, and Pareto-optimal. By Proposition 1, $f$ is dictatorial. Let agent 1 be the dictator. Fix $\theta^1 \in \Theta^1$, and let $\mathcal{K}_2 = \mathcal{K} \setminus \{\text{top}(\theta^1)\}$. Now let $f_2$ be an SCF which is defined for the set of agents $N_2 = N \setminus \{1\}$ and the set $\mathcal{K}_2$ such that $\forall \tilde{\theta}^{-1} \in \Theta^{-1}, \forall i \in N_2, f_2^i(\tilde{\theta}^{-1}) = f^i(\theta^1, \tilde{\theta}^{-1})$. Since $f$ is Pareto-optimal, and $f^1(\theta^1, \tilde{\theta}^{-1}) = \text{top}(\theta^1), \forall \tilde{\theta}^{-1} \in \Theta^{-1}$, $f_2$ is also Pareto-optimal. Since $f$ is strategyproof and nonbossy, no agent $i \in N_2$ can manipulate or be bossy at $(\theta^1, \tilde{\theta}^{-1})$ for any $\tilde{\theta}^{-1} \in \Theta^{-1}$. Therefore, $f_2$ is strategyproof and nonbossy. Thus, by Proposition 1, $f_2$ is dictatorial. (If $\mathcal{K}_2$ is a singleton, we can use Proposition 4 in Papai (1996).) Let agent 2 be the dictator for $f_2$. Note that the identity of the dictator for $f_2$ may only depend on $\theta^1$. Now fix $\theta^2 \in \Theta^2$, etc. Repeating the same argument for $n = 2, \ldots, n-1$, this proves that $f$ is a sequential choice function such that $\forall \theta, \tilde{\theta} \in \Theta, \sigma^1_\theta = \sigma^1_{\tilde{\theta}}$, and, for $j = 2, \ldots, n-1, \sigma^j(\theta)$ depends only on $\theta^{\sigma^i_\theta}, i = 1, \ldots, j-1, \forall \theta \in \Theta$, where $\sigma_\theta$ is an s-hierarchy associated with $f$ at $\theta$.

Now fix $\sigma_\theta \in \Sigma(N), \forall \theta \in \Theta$, such that $\sigma_\theta$ is an s-hierarchy associated with $f$ at $\theta$. Let $\sigma^1_\theta = 1, \forall \theta \in \Theta$, and fix $\theta^1 \in \Theta^1$. Let $\tilde{\theta}^1 \in \Theta^1$ be such that, $f^1(\theta) = f^1(\tilde{\theta}^1, \theta^{-1})$, where $\theta^1 \neq \tilde{\theta}^1$. (For example, let $\tilde{\theta}^1 \in r(f(\theta), \theta^1)$.) Note that $\theta^{-1} \in \Theta^{-1}$ is arbitrary. Now let $\sigma^2_\theta = i$ and $\sigma^2_{(\tilde{\theta}^1,\theta^{-1})} = j$. Suppose $i \neq j$. Since $f$ is a sequential choice function, $f^i(\theta) = c^i(\mathcal{K} \setminus \{\text{top}(\theta^1)\}, \theta^i)$ and $f^j(\tilde{\theta}^1, \theta^{-1}) = c^j(\mathcal{K} \setminus \{\text{top}(\tilde{\theta}^1)\}, \theta^j)$. However, $\text{top}(\theta^1) = \text{top}(\tilde{\theta}^1)$. Therefore, $f^j(\tilde{\theta}^1, \theta^{-1}) = c^j(\mathcal{K} \setminus \{\text{top}(\theta^1)\}, \theta^j)$. Now suppose, without loss of generality, that $\theta^i$ and $\theta^j$ satisfy $c^i(\mathcal{K} \setminus \{\text{top}(\theta^1)\}, \theta^i) = c^j(\mathcal{K} \setminus \{\text{top}(\theta^1\}), \theta^j)$. Then $f^i(\theta) = f^i(\tilde{\theta}^1, \theta^{-1})$ is not feasible, which violates $f$'s nonbossiness. Therefore, $i = j$, and thus $\forall \theta^1, \tilde{\theta}^1 \in \Theta^1$ such that $f^1(\theta) = f^1(\tilde{\theta}^1, \theta^{-1})$ implies that $\sigma^2_\theta = \sigma^2_{\tilde{\theta}}$, if $f^1(\theta) = f^1(\tilde{\theta})$, since $\sigma^2_\theta$ depends only on $\theta^1$ and $\sigma_\theta^2$ depends only on $\tilde{\theta}^1$. Repeating the same argument for $j = 3, \ldots, n$, we get that $f$ is a dictatorial sequential choice function.□

Note that if bossiness is allowed, then a strategyproof and Pareto-optimal SCF need not be dictatorial, as long as there are at least three agents. Thus, in general, when designing Pareto-optimal and strategyproof mechanisms for allocating heterogeneous objects, one may chose between dictatorial and bossy mechanisms. If there are only two agents, however, then Lemma 2 implies, together with Proposition 1, that any strategyproof and Pareto-optimal SCF is dictatorial. We give an example of a nondictatorial, strategyproof, and Pareto-optimal SCF below.

**Example 1** [7] *A nondictatorial, strategyproof, and Pareto-optimal SCF where $n = 3$.*

Let $f$ be a sequential choice function. Define $\tilde{\Theta} = \{\theta \in \Theta \mid$ if $\sigma_\theta = (1,2,3), f^3(\theta) = 0$ and if $\sigma_\theta = (2,1,3), f^3(\theta) = 0\}$, where $\sigma_\theta$ is an s-hierarchy associated with $f$ at $\theta$. Now fix $p \in \mathcal{K}, p \neq K, p \neq 0$. Let $\sigma_\theta = (1,2,3), \forall\theta \notin \tilde{\Theta}$ and $\forall\theta \in \tilde{\Theta}$ if $\theta^3 \in (p)$. Otherwise, let $\sigma_\theta = (2,1,3)$. Clearly, $f$ is Pareto-optimal, since it is a sequential choice function. It is nondictatorial, since, for example,

$$f\begin{pmatrix} p & p & p \\ K\setminus\{p\} & K\setminus\{p\} & \end{pmatrix} = (p, K\setminus\{p\}, 0),$$

and

$$f\begin{pmatrix} p & p & K\setminus\{p\} \\ K\setminus\{p\} & K\setminus\{p\} & \end{pmatrix} = (K\setminus\{p\}, p, 0).$$

To see that $f$ is strategyproof, note that agents 1 and 2 cannot affect the ordering at any profile, and that agent 3 can only affect the ordering when she is indifferent. This example works with any number of objects such that $k \geq 2$, and can easily be generalized to more than three agents.□

The above defined SCF is bossy. In particular, agent 3 is bossy at some profiles where she does not get any object, for example, at the above displayed two profiles.

# 4 Characterization of Strategyproof, Strongly Nonbossy, and Pareto-optimal Social Choice Functions

A sequntial choice mechanism with a single hierarchy, which we call a *serial dictatorship*, is a mechanism in which the agents get their favorite allocation from a feasible set (the remaining objects), according to a predetermined order. That is, the outcomes of a serial dictatorship correspond to a decentralized mechanism in which the agent who is ranked

[7] A similar example is provided in Satterthwaite and Sonnenschein (1981, Endnote 2), in the context of divisible goods. This is a very natural example of a nondictatorial and bossy mechanism, where an agent, who is a "loser" at certain profiles, gets to alternate the dictators at those profiles.

first chooses her favorite allocation from the fixed set of objects $K$, then the second agent chooses her favorite allocation from the remaining objects, etc, until all the objects are taken, or until we get to the last agent, whichever happens first. Note that since the first agent gets to "choose" from the set of all the objects, and all the subsequent agents "choose" from all the objects available after the higher ranked agents made their choices, these SCF's are Pareto-optimal. This contrasts with the observation of Satterthwaite and Sonnenschein (1981) that serial dictatorships violate Pareto-optimality, which they demonstrate with an example of a production economy. Since we do not consider production, a serial dictatorship is Pareto-optimal in our framework. It is also easy to verify that a serial dictatorship is strategyproof and nonbossy.

Similarly to Satterthwaite and Sonnenschein (1981), we examine which additional requirements, if imposed on an SCF, would imply that it is a serial dictatorship. It turns out that a mild strengthening of nonbossiness, which we call *strong nonbossiness*, is enough to constrain the choice of SCF's to serial dictatorships, when required in addition to strategyproofness and Pareto-optimality. Strong nonbossiness means that if an agent deviates at some profile, with the result that the extra objects that she obtains (if any) are unassigned at the given profile, and the objects that she loses (if any) remain unassigned at the new profile, then the other agents' allocations remain unchanged. In other words, strong nonbossiness requires that if an agent's action does not affect the others through the feasibility constraints then it should not affect the other agents at all. Clearly, strong nonbossiness implies nonbossiness, but bossiness does not imply strong nonbossiness, even if strategyproofness is also required. It can also be shown (analogously to Lemma 2) that a Pareto-optimal SCF is strongly nonbossy if there are only two agents.

**Definition 13** An SCF $f$ is *strongly nonbossy* if $\forall i \in N, \forall \theta \in \Theta$, and $\forall \tilde{\theta}^i \in \Theta^i$such that $f^i(\theta) \bigcap f^j(\tilde{\theta}^i, \theta^{-i}) = \emptyset$ and $f^i(\tilde{\theta}^i, \theta^{-i}) \bigcap f^j(\theta) = \emptyset, \forall j \in N, j \neq i$, we have $f^j(\theta) = f^j(\tilde{\theta}^i, \theta^{-i})$, $\forall j \in N, j \neq i$.

For the following definition, let the null object be defined as the empty set, i.e., let $0 = \emptyset$.

**Definition 14** An SCF $f$ is a *serial dictatorship* if $\exists \sigma \in \Sigma(N)$ such that $\forall \theta \in \Theta$, $f^{\sigma^1}(\theta) = c^{\sigma^1}(\mathcal{K}, \theta^{\sigma^1}) = \text{top}(\theta^{\sigma^1})$, and for $j \in N \setminus \{1\}, f^{\sigma^j}(\theta)$ are defined recursively by $f^{\sigma^j}(\theta) = c^{\sigma^j}(\mathcal{K} \setminus \bigcup_{i=1}^{j-1}\{f^{\sigma^i}(\theta)\}, \theta^{\sigma^j})$. We then call $\sigma$ the *d-hierarchy* associated with $f$.

Now we are ready to prove the characterization theorem.

**Proposition 4** *An SCF $f$ is strategyproof, strongly nonbossy, and Pareto-optimal if, and only if, it is a serial dictatorship.*

*Proof:* It is easy to check that a serial dictatorship is strategyproof, strongly nonbossy, and Pareto-optimal. In order to prove that a strategyproof, strongly nonbossy, and Pareto-optimal SCF is a serial dictatorship, we need to introduce some definitions. The proof will proceed by several lemmas.

**Definition 15** An SCF $f$ is *multihierarchical* if $\exists \sigma : \Theta \mapsto \Sigma(N)$ such that $\forall i, j \in N$, if $\sigma_\theta(i) < \sigma_\theta(j)$ then $U(f^i(\theta), \theta^i) > U(f^j(\theta), \theta^i)$, unless $f^i(\theta) = f^j(\theta) = 0$. We then call $\sigma_\theta$ an *m-hierarchy* associated with $f$ *at* $\theta$.

Therefore, if $f$ is multihierarchical then there exists a "hierarchy" of the agents for each profile, not necessarily the same for each profile, such that each agent prefers her allocation at that profile to the allocation of all the agents at the same profile who rank lower than she in the hierarchy for that profile. Thus, loosely speaking, if there is a "conflict" among agents at some profile then it is resolved according to the hierarchy at that profile.

**Definition 16** The *top set* $T(j, \theta)$ for each agent $j$ and profile $\theta$ contains the sets of objects that $j$ prefers to her allocation at that profile, given some SCF $f$. That is, $T(j, \theta) = \{p \in \mathcal{K} \mid U(p, \theta^j) > U(f^j(\theta), \theta^j)\}, \forall j \in N, \forall \theta \in \Theta$.

Clearly, $\forall \theta \in \Theta, \forall j \in N, 0 \notin T(j, \theta)$, since the objects need not be assigned.

**Definition 17** Given an SCF $f$, agent $i$ *beats* agent $j$ at $\theta$, if $f^i(\theta) \in T(j, \theta)$. This relationship is denoted by $B(\theta)$. That is, if $i$ beats $j$ at $\theta$, then we write $iB(\theta)j$.

**Lemma 5** *A Pareto-optimal SCF is multihierarchical.*

*Proof:* Let $f$ be a Pareto-optimal SCF. Then $\forall \theta \in \Theta, B(\theta)$ is acyclic for $f$. That is, $\forall \theta \in \Theta$, if $i_1 B(\theta) i_2 B(\theta) \cdots B(\theta) i_t$ for $i_l \in N, l = 1, \ldots, t$, $2 \leq t \leq n$, then $\neg(i_t B(\theta) i_1)$.[8] This implies that $\forall \theta \in \Theta, \exists \sigma \in \Sigma(N)$ such that $\forall i, j \in N$ if $iB(\theta)j$ then $\sigma_\theta(j) > \sigma_\theta(i)$. Then $\forall i, j \in N, \forall \theta \in \Theta, \sigma_\theta(j) > \sigma_\theta(i)$ implies that $\neg(jB(\theta)i)$, which in turn implies that $U(f^i(\theta), \theta^i) \geq U(f^j(\theta), \theta^i)$. But $f^i(\theta) \neq f^j(\theta)$, unless $f^i(\theta) = f^j(\theta) = 0$, since $f(\theta)$ is not feasible otherwise. Thus, $U(f^i(\theta), \theta^i) > U(f^j(\theta), \theta^i), \forall i, j \in N, \forall \theta \in \Theta$ if $\sigma_\theta(j) > \sigma_\theta(i)$, unless $f^i(\theta) = f^j(\theta) = 0$. Therefore, $\forall \theta \in \Theta, \sigma_\theta$ is an m-hierarchy associated with $f$ at $\theta$, and thus $f$ is multihierarchical.$\square$

Let $\theta^i \in \begin{pmatrix} a \\ b \end{pmatrix}$ denote some preferences of agent $i$ such that $a$ is ranked first, $y$ is ranked second, and the rest of the preferences is arbitrary. We use a similar notation for profiles. For example, $\theta \in \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ if $\theta^1 \in \begin{pmatrix} a \\ b \end{pmatrix}$ and $\theta^2 \in \begin{pmatrix} c \\ d \end{pmatrix}$. Furthermore, we write $f\begin{pmatrix} a & c \\ b & d \end{pmatrix} = x$ to indicate that $f$ assigns outcome $x$ to all profiles in $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

**Lemma 6** *For every strategyproof, strongly nonbossy, and Pareto-optimal SCF there exists a single m-hierarchy that is associated with it at every profile.*

[8]The logic symbol $\neg$ means 'not' in this study.

*Proof:* Let $f$ be strategyproof, strongly nonbossy, and Pareto-optimal. For Steps 1–3, fix $i, j \in N$ and $\theta, \tilde{\theta} \in \Theta$ such that $f^i(\theta) = K, iB(\theta)j$, and $jB(\tilde{\theta})i$. By Pareto-optimality, $\forall i \in N, \exists \theta \in \Theta$ such that $f^i(\theta) = K$. If there do not exist $j$ and $\tilde{\theta}$ such that $iB(\theta)j$, and $jB(\tilde{\theta})i$, where $f^i(\theta) = K$ then the lemma holds. Let $f(\theta) = x$ and $f(\tilde{\theta}) = y$.

**Step 1:** If $iB(\tilde{\theta})j$ such that $f^i(\theta) = K$ and $jB(\tilde{\theta})i$ for some $j \in N$ and $\tilde{\theta} \in \Theta$ then $f^j(\tilde{\theta}) \neq K$.

Suppose $f^j(\theta) = K$. Let $\bar{\theta}^i, \bar{\theta}^j \in \begin{pmatrix} K \\ 0 \end{pmatrix}$. Let $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i,j\}$. Given that $iB(\theta)j$, $f$'s monotonicity implies that $f(\bar{\theta}) = x$. However, since $jB(\tilde{\theta})i$, monotonicity also implies that $f(\bar{\theta}) = x$. Since $x = y$ contradicts feasibility, $f^j(\tilde{\theta}) \neq K$.

**Step 2:** If $jB(\tilde{\theta})i$ for some $i, j \in N, \tilde{\theta} \in \Theta$ then $\exists \bar{\theta} \in \Theta$ with $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i,j\}$ such that $jB(\bar{\theta})i$.

Suppose that $\forall \bar{\theta} \in \Theta$ such that $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i,j\}, \neg(jB(\bar{\theta})i)$. Let $\bar{\bar{\theta}}^i \in \begin{pmatrix} y^j \\ y^i \\ 0 \end{pmatrix}$ and $\bar{\bar{\theta}}^l \in \begin{pmatrix} y^l \\ 0 \end{pmatrix}, \forall l \in N \setminus \{i\}$. Then $f(\bar{\bar{\theta}}) = y$ by monotonicity, given that $jB(\tilde{\theta})i$ and $y^j \in T(i, \tilde{\theta}^{-i})$. If $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i,j\}$ then Pareto-optimality and feasibility imply that $f^i(\bar{\bar{\theta}}^i, \bar{\bar{\theta}}^j, \bar{\theta}^{-i,j}) = y^j$, given that $\neg\left(jB(\bar{\bar{\theta}}^i, \bar{\bar{\theta}}^j, \bar{\theta}^{-i,j})i\right)$. However, since $f$ is strongly nonbossy, Pareto-optimality implies that $f^i(\bar{\bar{\theta}}^i, \bar{\bar{\theta}}^j, \bar{\bar{\theta}}^{\bar{L}}, \bar{\theta}^L) = y^i, \forall L \subseteq N \setminus \{i,j\}$, where $\bar{L} = N \setminus (\{i,j\} \bigcup L)$. For $L = N \setminus \{i,j\}$ we get that $f^i(\bar{\bar{\theta}}^i, \bar{\bar{\theta}}^j, \bar{\theta}^{-i,j}) = y^i$. This implies that $y^i = y^j$, so that $y^i = y^j = 0$, given the feasibility constraints. However, $y^j \in T(i, \tilde{\theta}^{-i})$ implies that $y^j \neq 0$, which is a contradiction. Therefore, if $jB(\tilde{\theta})i$ for some $i, j \in N$, and $\tilde{\theta} \in \Theta$ then $\exists \bar{\theta} \in \Theta$ with $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i,j\}$ such that $jB(\bar{\theta})i$.

**Step 3:** If $jB(\bar{\theta})i$ such that $f^j(\bar{\theta}) \neq K$ and $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i,j\}$ then $\exists \bar{\bar{\theta}} \in \Theta$ such that $jB(\bar{\bar{\theta}})i$ and $f^j(\bar{\theta}) = K$.

Let $f(\bar{\theta}) = z$ and let $i = 1, j = 2$. By assumption, $z^2 \neq K$, and since $2B(\bar{\theta})1, z^2 \neq 0$. By Pareto-optimality, $z^l = 0, \forall l \in N \setminus \{1,2\}$. By monotonicity, $f\begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ z^1 & 0 & & & \end{pmatrix} = z = (z^1, z^2, 0, \ldots, 0)$. Now consider some profile in $\begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ 0 & 0 & & & \end{pmatrix}$. If $z^1 \neq 0$ then Pareto-optimality implies that either agent 1 or agent 2 gets $z^2$ at this profile. If agent 1 gets $z^2$ then she can manipulate at $\begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ z^1 & 0 & & & \end{pmatrix}$ via $\begin{pmatrix} z^2 \\ 0 \end{pmatrix}$. Therefore, Pareto-optimality yields $f\begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ 0 & 0 & & & \end{pmatrix} = (0, z^2, 0, \ldots, 0)$. Then

$$f\begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ K & 0 & & & \\ 0 & & & & \end{pmatrix}$$

$= (0, z^2, 0, \ldots, 0)$, since the other Pareto-optimal outcome, $(z^2, 0, \ldots, 0)$, would enable agent 1 to manipulate at $\begin{pmatrix} z^2 & z^2 & 0 & \cdots & 0 \\ 0 & 0 & & & \end{pmatrix}$ via $\begin{pmatrix} z^2 \\ K \\ 0 \end{pmatrix}$. Now consider some profile in $\begin{pmatrix} z^2 & K & 0 & \cdots & 0 \\ K & z^2 & & & \\ 0 & 0 & & & \end{pmatrix}$. There are two Pareto-optimal outcomes at these profiles, $(z^2, 0, \ldots, 0)$ and $(0, K, 0, \ldots, 0)$, given the feasibility constraints. If the outcome is the former then agent 2 can manipulate at these profiles via $\begin{pmatrix} z^2 \\ 0 \end{pmatrix}$. Therefore,
$f^2\begin{pmatrix} z^2 & K & 0 & \cdots & 0 \\ K & z^2 & & & \\ 0 & 0 & & & \end{pmatrix} = K$. Letting one of these profiles be $\bar{\bar{\theta}}$, we get that $f^j(\bar{\bar{\theta}}) = f^2(\bar{\bar{\theta}}) = K$, and $K \in T(i, \bar{\bar{\theta}})$ implies that $jB(\bar{\bar{\theta}})i$, as desired.

**Step 4:** If $iB(\theta)j$ for some $i, j \in N$ and $\theta \in \Theta$ such that $f^i(\theta) = K$ then $\forall \tilde{\theta} \in \Theta, \neg(jB(\tilde{\theta})i)$.

This step follows from Steps 1-3. Suppose $iB(\theta)j$ for some $i, j \in N$ and $\theta \in \Theta$ such that $f^i(\theta) = K$ and $jB(\tilde{\theta})i$ for some $\tilde{\theta} \in \Theta$. Since $jB(\tilde{\theta})i, \exists \bar{\theta}$ with $\bar{\theta}^l \in (0), \forall l \in N \setminus \{i, j\}$ such that $jB(\bar{\theta})i$, by Step 2. Then $\exists \bar{\bar{\theta}} \in \Theta$ such that $f^j(\bar{\bar{\theta}}) = K$ and $jB(\bar{\bar{\theta}})i$, by Step 3. However, this contradicts the assumption that $iB(\theta)j$ where $f^i(\theta) = K$, by Step 1. Therefore, $\forall i, j \in N$ if $iB(\theta)j$ for some $\theta \in \theta$ such that $f^i(\theta) = K$ then $\forall \tilde{\theta} \in \Theta, \neg(jB(\tilde{\theta})i)$.

**Step 5:** There exists $\sigma \in \Sigma(N)$ such that $\sigma$ is an m-hierarchy associated with $f$ at each profile.

Let $\theta_{[1]} \in \begin{pmatrix} K & \cdots & K \\ 0 & \cdots & 0 \end{pmatrix}$. Then $\exists i \in N$ such that $f^i(\theta_{[1]}) = K$, by Pareto-optimality, and then $f^j(\theta_{[1]}) = 0, \forall j \in N \setminus \{i\}$, given the feasibility constraints. Let $i = 1$. Now let $\theta_{[2]} \in \begin{pmatrix} 0 & K & \cdots & K \\ & 0 & \cdots & 0 \end{pmatrix}$. Then $\exists i' \in N \setminus \{1\}$ such that $f^{i'}(\theta_{[2]}) = K$, by Pareto-optimality, and so $f^j(\theta_{[2]}) = 0, \forall j \in N \setminus \{i'\}$, given the feasibility constraints. Let $i' = 2$. Continuing in the same manner, we get an ordering of the agents, $\sigma = (1, \ldots, n)$. Now fix $i, j \in N$. If $i < j$ then $iB(\theta_{[i]})j$ and $f^i(\theta_{[i]}) = K$. Then Step 4 implies that $\forall \theta \in \Theta, \neg(jB(\theta)i)$. Thus, $\forall \theta \in \Theta, U(f^i(\theta), \theta^i) > U(f^j(\theta), \theta^i)$, unless $f^i(\theta) = f^j(\theta) = 0$. Therefore, $\sigma = (1, \ldots, n)$ is an m-hierarchy associated with $f$ at each profile $\theta$.$\square$

Let $o(i, \theta^{-i}) = \{x \in \mathcal{X} \mid \exists \theta^i \in \Theta^i \text{such that} f(\theta) = x\}$ denote agent $i$*'s option set* (for $f$) at profile $\theta$. Let $o^i(i, \theta^{-i}) = \{p \in \mathcal{K} \mid \exists \theta^i \in \Theta^i \text{such that} f^i(\theta) = p\}$. That is, $o^i(i, \theta^{-i})$ is the set of allocations that agent $i$ can get by deviating her messages when the other agents' report is $\theta^{-i}$. Clearly, $0 \in o^i(i, \theta^{-i}), \forall i \in N, \forall \theta \in \Theta$.

**Lemma 7** *If an SCF is strategyproof, strongly nonbossy, and Pareto-optimal such that there exists a single m-hierarchy, $\sigma$, associated with it at each profile then it is a serial dictatorship with d-hierarchy $\sigma$.*

*Proof:* Let $f$ be strategyproof, strongly nonbossy, and Pareto-optimal such that $\sigma = (1,\ldots,n)$ is an m-hierarchy associated with it at each profile $\theta$.

**Step 1:** If $p \in T(i,\theta)$ then $\exists j \in N, j < i$, such that $p \bigcap f^j(\theta) \neq \emptyset$.

Fix $i \in N, \theta \in \Theta$, and $p \in \mathcal{K}, p \neq 0$. Suppose that $p \in T(i,\theta)$ for some $i \in N$ and $\forall j \in N \setminus \{i\}, j < i, p \bigcap f^j(\theta) = \emptyset$. Then Pareto-optimality implies that $\exists j \in N \setminus \{i\}$ such that $p \bigcap f^j(\theta) \neq \emptyset$. Suppose there are $t \geq 1$ such agents, $j_1,\ldots,j_t$, i.e., for $l = 1,\ldots,t, p \bigcap f^{j_l} \neq \emptyset$ such that $j_l > i$. Let $f(\theta) = x$. Let $\bar{\theta}^j \in \begin{pmatrix} x^j \\ 0 \end{pmatrix}, \forall j \in N \setminus \{i\}$ and let $\bar{\theta}^i \in \begin{pmatrix} p \\ x^i \\ 0 \end{pmatrix}$. Then monotonicity implies that $f(\bar{\theta}) = x$. By strong nonbossiness and Pareto-optimality, $f(\bar{\theta}^i, \bar{\theta}^J, 0, \ldots, 0) = (x^i, x^J, 0, \ldots, 0)$, where $J = \{j_1,\ldots,j_t\}, x^J = (x^{j_1},\ldots,x^{j_t})$ and 0 denotes a strategy in (0). For simplicity, let us ignore all $j \notin J, j \neq i$ for the rest of this proof, since their strategies will be kept the same (a strategy in (0)) and therefore they won't play any role. Thus, we have $f\begin{pmatrix} p & x^{j_1} & \cdots & x^{j_t} \\ x^i & 0 & \cdots & 0 \\ 0 & & & \end{pmatrix} = (x^i, x^{j_1}, \ldots, x^{j_t})$. Now consider a profile in $\begin{pmatrix} p & x^{j_1} & \cdots & x^{j_1} \\ x^{j_1} & 0 & \cdots & 0 \\ 0 & & & \end{pmatrix}$. Since $\neg(j_1 B(\theta) i), \forall \theta \in \Theta, j_1$ cannot get $x_{j_1}$ at this profile, given that $p \bigcap x^{j_1} \neq 0$. Then, by Pareto-optimality, agent $i$ gets either $p$ or $x^{j_1}$. If $i$ gets $p$ then she can manipulate at $\begin{pmatrix} p & x^{j_1} & \cdots & x^{j_t} \\ x^i & 0 & \cdots & 0 \\ 0 & & & \end{pmatrix}$ via $\begin{pmatrix} p \\ x^{j_1} \\ 0 \end{pmatrix}$. Therefore, agent $i$ gets $x^{j_1}$, and thus Pareto-optimality implies that $f\begin{pmatrix} p & x^{j_1} & \cdots & x^{j_t} \\ x^{j_1} & 0 & \cdots & 0 \\ 0 & & & \end{pmatrix} = (x^{j_1}, 0, x^{j_2}, \ldots, x^{j_t})$. By monotonicity, (or Pareto-optimality and nonbossiness), we get $f\begin{pmatrix} p & 0 & x^{j_2} & \cdots & x^{j_t} \\ x^{j_1} & & 0 & \cdots & 0 \\ 0 & & & & \end{pmatrix} = (x^{j_1}, 0, x^{j_2}, \ldots, x^{j_t})$. Now we can continue by replacing iteratively $i$'s strategy with $\begin{pmatrix} p \\ x^{j_2} \\ 0 \end{pmatrix}, \begin{pmatrix} p \\ x^{j_3} \\ 0 \end{pmatrix}$, etc. When we get to $\begin{pmatrix} p \\ x^{j_{t-1}} \\ 0 \end{pmatrix}$, we get $f\begin{pmatrix} p & 0 & \cdots & 0 & x^{j_t} \\ x^{j_{t-1}} & & & & 0 \\ 0 & & & & \end{pmatrix} = (x^{j_{t-1}}, 0, \ldots, 0, x^{j_t})$. Then

$f\left(\begin{array}{cccc} p & 0 & \cdots & 0 & x^{j_t} \\ x^{j_t} & & & & 0 \\ 0 & & & & \end{array}\right) = (p, 0, \ldots, 0)$, since $\neg(j_t B(\theta) i), \forall \theta \in \Theta$, and so $j_t$ cannot get $x^{j_t}$. Then, agent $i$ gets $p$, by Pareto-optimality. However, in this case, agent $i$ can manipulate at $\left(\begin{array}{ccccc} p & 0 & \cdots & 0 & x^{j_t} \\ x^{j_{t-1}} & & & & 0 \\ 0 & & & & \end{array}\right)$ via $\left(\begin{array}{c} p \\ x^{j_t} \\ 0 \end{array}\right)$, which contradicts $f$'s strategyproofness. Therefore, $\forall i \in N, \forall p \in \mathcal{K}, p \neq 0, \forall \theta \in \Theta$ if $p \in T(i, \theta)$ then $\exists j \in n, j < i$ such that $p \bigcap f^j(\theta) \neq \emptyset$.

**Step 2:** $f$ is a serial dictatorship where $\sigma$ is the d-hierarchy associated with $f$.

For this step, set $0 = \emptyset$. Fix $i \in N$ and $p \in \mathcal{K}$. Let $\theta^i$ be such that $\text{top}(\theta^i) = p$, and suppose that $\theta^{-i} \in \Theta^{-i}$ is such that $\forall j \in N \setminus \{i\}, j < i, p \bigcap f^j(\theta) = \emptyset$. Then $p \notin T(i, \theta)$, by Step 1, and so $f^i(\theta) = p$. This proves that

$$\forall i \in N, \forall p \in \mathcal{K}, \forall \theta \in \Theta, \text{ if } p \bigcap f^j(\theta) = \emptyset, \forall j \in N \setminus \{i\}, j < i, \text{ then } p \in o^i(i, \theta^{-i}). \quad (3)$$

Since $\forall j \in N \setminus \{1\}, j > 1$, we have $p \in o^1(1, \theta^{-1}), \forall p \in \mathcal{K}, \forall \theta^{-1} \in \Theta^{-1}$, which implies that $o^1(1, \theta^{-1}) = \mathcal{K}$. Now fix $i \in N \setminus \{1\}$. Suppose $p \bigcap f^1(\theta) \neq \emptyset$ and $p \in o^i(i, \theta^{-i})$, for some $\theta \in \Theta$ and $p \in \mathcal{K}$. Then $\exists \tilde{\theta}^i \in \Theta^i$ such that $f^i(\tilde{\theta}^i, \theta^{-i}) = p$. Then $f^1(\tilde{\theta}^i, \theta^{-i}) \neq f^1(\theta)$, given the feasibility constraints. Clearly, if $f$ is strategyproof and nonbossy, then the outcome at every profile is the best option at that profile for each agent. That is, $\forall \theta \in \Theta, \forall i \in N, f^i(\theta) = c^i(o(i, \theta^{-i}), \theta^i)$. Since $o^1(1, (\tilde{\theta}^i, \theta^{-i,1})) = \mathcal{K}$, this implies that $f^1(\tilde{\theta}^i, \theta^{-i}) = f^1(\theta) = c^1(\mathcal{K}, \theta^1)$, which is a contradiction. Then $\forall i \in N, \forall p \in \mathcal{K}, \forall \theta \in \Theta$, if $p \bigcap f^1(\theta) \neq \emptyset, p \notin o^2(2, \theta^{-2})$. This implies, together with (3), that $o^i(i, \theta^{-i}) = \mathcal{K} \setminus \{f^1(\theta)\}, \forall \theta \in \Theta$. Now fix $i \in N \setminus \{1, 2\}$. Suppose $p \bigcap f^2(\theta) \neq \emptyset$ and $p \in o^i(i, \theta^{-i})$, for some $\theta \in \Theta, p \in \mathcal{K}$. Then a similar argument to the one applied to agent 1 above shows that this is a contradiction, and we can imply that $o^3(3, \theta^{-3}) = \mathcal{K} \setminus \{(f^1(\theta)\} \bigcup \{f^2(\theta)\}$, using (3). Continuing iteratively, we get that $\forall \theta \in \Theta, \forall i \in N \setminus \{1\}, o^i(i, \theta^{-i}) = \mathcal{K} \setminus \bigcup_{l=1}^{i-1} \{f^l(\theta)\}$, where $o^1(1, \theta^{-1}) = \mathcal{K}$. Thus, we have $f^1(\theta) = c^1(\mathcal{K}, \theta^1)$ and for $i \in N \setminus \{1\}, f^i(\theta) = c^i(\mathcal{K} \setminus \bigcup_{l=1}^{i-1} \{f^l(\theta)\}, \theta^{-i}), \forall \theta \in \Theta$. Therefore, $f$ is a serial dictatorship such that the d-hierarchy associated with $f$ is $(1, \ldots, n)$.□

Proposition 4 follows immediately from the three lemmas.

It should be remarked that Satterthwaite and Sonnenschein's result (1981, Theorem 2) does not imply ours. Although they do not require Pareto-optimality, they impose a number of differentiabilty assumptions on the social choice function (which they call regularity) and assume that each agents' consumption set is convex. These assumptions do not apply to economies with indivisibilities.

It is interesting, however, to compare their sufficiency condition for serial dictatorships to ours. Their requirement is that the affect relation is *everywhere total* (in the following, ET), i.e., at each profile for any two agents at least one of them affects the other. Our result, therefore, looks surprising, since the strong nonbossiness condition rules out some affect relations under certain circumstances. Furthermore, a serial dictatorship in our context does not satisfy ET. To see this, take three agents, say agents 1, 2, and 3, such that $f^1(\theta) = p$ at some profile $\theta$, where $p \in \mathcal{K}, p \neq 0$, and $\theta^2, \theta^3 \in \begin{pmatrix} p \\ 0 \end{pmatrix}$. Since agent 1 beats both agents 2 and 3 at $\theta$, the d-hierarchy $\sigma$ associated with the serial dictatorship $f$ is such that $\sigma(1) < \sigma(2)$ and $\sigma(1) < \sigma(3)$. Now assume that $\sigma(2) < \sigma(3)$, so that $f^3(\theta) = 0$, by Pareto-optimality. Then agent 1 cannot affect agent 3 at this profile, since for each $\tilde{\theta}^1 \in \Theta^1$, if $f^1(\tilde{\theta}^1, \theta^{-1}) \bigcap p \neq \emptyset$ then, given the feasibility constraints, Pareto-optimality requires that $f^3(\tilde{\theta}^1, \theta^{-1}) = 0$. In addition, if $f^1(\tilde{\theta}^1, \theta^{-1}) \bigcap p = \emptyset$ then $f^3(\tilde{\theta}^1, \theta^{-1}) = 0$ again, since $\sigma(2) < \sigma(3)$, and $\theta^2 \in \begin{pmatrix} p \\ 0 \end{pmatrix}$. Notice, however, that it is Pareto-optimality that seems to be in conflict with ET. Indeed, if an SCF satisfies ET in our context, then it cannot be Pareto-optimal. (To check this, take a profile in which two agents' first choice is the null object.) Thus, the ET condition is too restrictive in our context. Pareto-optimality, however, is too restrictive in the Satterthwaite-Sonnenschein model, as they remark that for some standard convex and compact allocation possibility sets, the set of Pareto-optimal SCF's is empty. Although the two conditions are not necessarily compatible, they are essentially similar in their effects. To see this, note that Satterthwaite and Sonnenschein don't require any form of citizen sovereignty, that is, variation in the outcomes. Therefore, their Theorem 1 is consistent with an imposed mechanism,[9] which says that for a strategyproof, nonbossy, and regular mechanism, the affect relationship is acyclic at any profile, if the domain is some open set of utility functions. That is, an imposed mechanism which yields the same outcome at any profile, a mechanism for which no agent affects any other agent at any profile, would satisfy the theorem. Therefore, in order to get a serial dictatorship, they need to require some variation in the outcomes, and ET implies just that. In light of Proposition 2, our Pareto-optimality requirement has essentially a similar effect.

This still does not explain the sufficiency of strong nonbossiness. Remark that in our model, since the objects need not be allocated, and the value of any set of objects may be negative to an agent, Pareto-optimality requires that at some profiles not all the objects are allocated. Apparently, serial dictatorship can be avoided using this type of lack of "conflict," so that when some variation in the outcomes are ruled out in these "no conflict" situations, and that's what strong nonbossiness amounts to, the ordering of the agents induced by Pareto-optimality must be the same for all profiles, causing the mechanism to be a serial dictatorship.

# 5 Discussion

We presented two main results in this paper. Firstly, we proved that all strategyproof, nonbossy, and Pareto-optimal SCF's are dictatorial sequential choice functions. Secondly, we showed that all strategyproof, strongly nonbossy, and Pareto-optimal SCF's are serial

[9]This is pointed out in Muller and Satterthwaite (1986).

dictatorships. It is interesting to note that the concepts of strong nonbossiness and nonbossiness are identical on domains of high conflict (e.g., when the objects are desirable) if Pareto-optimality is also required, or when the contention for the object(s) is high due to the feasibility constraints (e.g., when a single object is being allocated). Thus, using a serial dictatorship may be necessary if the potential conflict of interests is severe. We remark that the two results are the same if there is only a single object, given that serial dictatorships and dictatorial sequential choice functions are also identical in this case.[10]

The results in this paper were established for strict preferences over allocations. We conjecture that they would also hold if weak preferences were admissable. Since an agent's choice from a given choice set may not be uniquely defined when weak preferences are allowed, this may lead to difficulties in defining Pareto-optimal SCF's.[11]

[10] See Papai (1996)

[11] For an illustration of this problem see, for example, Svensson (1994).

## References


Barbera, Salvador (1983), "Strategy-Proofness and Pivotal Voters: A Direct Proof of the Gibbard-Satterthwaite Theorem," *International Economic Review,* 24: 413-417.

Barbera, Salvador and Matthew O. Jackson (1995), "Strategy-Proof Exchange," *Econometrica,* 63: 51-87.

Barbera, Salvador and B. Peleg (1990), "Strategy-Proof Voting Schemes with Continuous Preferences," *Social Choice and Welfare* 7: 31-37.

Dasgupta, Partha, Peter Hammond and Eric S. Maskin (1978), "The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility," *Review of Economic Studies,* 46: 185-216.

Gibbard, Allan (1973), "Manipulation of Voting Schemes: A General Result," *Econometrica,* 41: 587-601.

Kalai, Ehud and Zvi Ritz (1980), "Characterization of the Private Alternatives Domains Admitting Arrow Social Welfare Functions," *Journal of Economic Theory,* 22: 23-36.

Laffont, Jean-Jacques and Eric S. Maskin (1982), "Nash and Dominant Strategy Implementation in Economic Environments," *Journal of Mathematical Economics,* 10: 17-47.

Le Breton, Michel and Arunava Sen (1995), "Strategyproofness and Decomposability: Strict Orderings," Discussion Paper No. 95-03, Indian Statistical Institute.

Muller, Eitan and Mark A. Satterthwaite (1977), "The Equivalence of Strong Positive Association and Strategy-Proofness," *Journal of Economic Theory,* 14: 412-418.

Muller, Eitan and Mark A. Satterthwaite (1986), "Strategyproofness: The Existence of Dominant-Strategy Mechanisms," in *Social Goals and Social Organizations: Volume in Memory of Elisha Pazner,* ed., L. Hurwicz, Cambridge University Press, New York.

Olson, Mark A. (1991), "The Assignment Problem: Theory and Experiments," Thesis, California Institute of Technology.

Papai, Szilvia (1996), "Strategyproof Allocation of a Single Object," Social Science Working Paper No. 936, California Institute of Technology.

Ritz, Zvi (1983), "Restricted Domains, Arrow Social Welfare Functions and Noncorruptible and Nonmanipulable Social Choice Correspondences: The Case of Private Alternatives," *Mathematical Social Sciences,* 4: 155-179.

Satterthwaite, Mark A. (1975) "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," *Journal of Economic Theory,* 10: 187-217.

Satterthwaite, Mark A. and Hugo Sonnenschein (1981), "Strategy-Proof Allocation Mechanisms at Differentiable Points," *Review of Economic Studies,* 48: 587-597.

Sprumont, Yves (1995), "Strategyproof Collective Choice in Economic and Political Environments, *Canadian Journal of Economics,* 28: 68-107.

Svensson, Lars-Gunnar (1994), "Queue Allocation of Indivisible Goods," *Social Choice and Welfare,* 11: 323-330.

Zhou, Lin (1991a), "Impossibility of Strategy-Proof Mechanisms in Economies with Pure Public Goods," *Review of Economic Studies,* 58: 107-119.

Zhou, Lin (1991b), "Inefficiency of Strategy-Proof Allocation Mechanisms in Pure Exchange Economies," *Social Choice and Welfare,* 8: 247-254.